AMS Short Course in Seattle, WA

AMS Short Course on Rigorous Numerics in Dynamics

This two-day course will take place on Monday and Tuesday, January 4 and 5, before the joint meeting actually begins. It is co-organized by **Jean-Philippe Lessard**, Université Laval, Québec, Canada, & **Jan Bouwe van den Berg**, VU University Amsterdam, Netherlands.

Nonlinear dynamics shape the world around us, from the harmonious movements of celestial bodies, via the swirling motions in fluid flows, to the complicated biochemistry in the living cell. Mathematically these beautiful phenomena are modelled by nonlinear dynamical systems, mainly in the form of ordinary differential equations (ODEs), partial differential equations (PDEs) and delay differential equations (DDEs). The presence of nonlinearities severely complicates the mathematical analysis of these dynamical systems, and the difficulties are even greater for PDEs and DDEs, which are naturally defined on infinite dimensional function spaces. With the availability of powerful computers and sophisticated software, numerical simulations have quickly become the primary tool to study the models. However, while the pace of progress increases, one may ask: just how reliable are our computations? Even for finite dimensional ODEs, this question naturally arises if the system under study is chaotic, as small differences in initial conditions (such as those due to rounding errors in numerical computations) yield wildly diverging outcomes. These issues have motivated the development of the field of rigorous numerics in dynamics.

Rigorous numerics draws inspiration from the ideas in scientific computing, numerical analysis and approximation theory.

It is well suited to a short course, as it concerns recent research progress in applied mathematics, while only a basic mathematical background is required to appreciate the striking interplay between theory, computations and applications.

Dynamics and Chaos For Maps and The Conley Index

Sarah Day, The College of William & Mary

Sarah Day

Discrete-time dynamical systems modeled by iteration of continuous maps exhibit a wide variety of interesting behaviors. One illustrative example is the one-dimensional logistic model. For the logistic model, chaotic dynamics may be proven via a topological conjugacy onto an appropriate subshift of finite type, a symbolic system for which a proof of chaos is attainable. Analysis and proofs of dynamics for other discrete-time models, especially in dimensions larger than one, often prove to be more challenging. In this course, we examine methods for constructing *outer approximations*, finite representations of discrete-time models that are amenable to computational studies and computer-assisted proofs. These methods rely heavily on *Conley index theory*, an algebraic topological generalization of Morse Theory. Both theory and algorithms will be presented in this course. Studies of models including pulse-coupled oscillator systems and the infinite-dimensional Kot–Schaffer model from ecology will serve as illustrations of the methods.

References

[1] S. Day, O. Junge, K. Mischaikow, *Toward automated chaos verification*, In EQUADIFF 2003, pages 157–162, World Sci. Publ., Hackensack, NJ, 2005.

[2] Z. Arai., W. Kalies, H. Kokubu, K. Mishchaikow, H. Oka, P. Pilarczyk, A database schema for the analysis of global dynamics of multiparameter systems, *SIAM J. Appl. Dyn. Syst.*, 8(3):757–789, 2009.

[3] S. Day, R. Frongillo, R. Treviño, Algorithms for rigorous entropy bounds and symbolic dynamics, *SIAM J. Appl. Dyn. Syst.*, 7(4):1477–1506, 2008.

[4] T. Kaczynski, K. Mischaikow, M. Mrozek, Computational homology, volume 157 of *Applied Mathematical Sciences*, Springer-Verlag, New York, 2004.

[5] K. Mischaikow and M. Mrozek, Conley index, In *Handbook of Dynamical Systems*, Vol. 2, pages 393–460, North-Holland, Amsterdam, 2002.

Delay Differential Equations and Continuation

Jean-Philippe Lessard, Université Laval

Université Laval/Marc Robitaille.

Jean-Philippe Lessard

An intriguing feature of the study of nonlinear delay differential equations (DDEs) is that progress in understanding their dynamics has been slow and has involved deep mathematical ideas. This is perhaps not surprising as a large class of DDEs naturally give rise to infinite dimensional nonlinear dynamical systems. Even for the simplest-looking DDEs, many fundamental dynamical questions remain open. In particular, the study of the global dynamics of Wright's equation defined by

$$y'(t) = -\alpha y(t-1)[1 + y(t)], \qquad \alpha \in \mathbb{R} \tag{1}$$

has been the subject of active research for sixty years. In 1955, E. M. Wright considered this equation because of its role in the distribution of prime numbers [6]. A conjecture (stated by Jones in 1962 [7]) asserts that (1) has a unique slowly oscillating periodic solution for all $\alpha > \pi/2$; i.e., a periodic solution that oscillates around 0, spending more than one unit of time (per period) on either side of 0.

In this lecture we show how ideas from rigorous computations can be used to study the dynamics of DDEs. In particular, with the help of Fourier series, we introduce a continuation method to compute global branches of periodic solutions of DDEs. We discuss progress on the study of the long-standing above mentioned conjecture, as discussed in [2].

References

[1] R. D. Nussbaum, *Functional differential equations*, Handbook of dynamical systems, Vol. 2, 461–499, North-Holland, Amsterdam, 2002.

[2] J.-P. Lessard, *Recent advances about the uniqueness of the slowly oscillating periodic solutions of Wright's equation*, Journal of Differential Equations, 248 (5): 992–1016, 2010.

[3] G. Kiss and J.-P. Lessard, *Computational fixed point theory for differential delay equations with multiple time lags*, Journal of Differential Equations, 252 (4): 3093–3115, 2012

[4] _____, *Rapidly and slowly oscillating periodic oscillations of a delayed van der Pol oscillator*, Submitted, 2015.

[5] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel and H.-O. Walther, *Delay equations. Functional, complex, and nonlinear analysis*, Applied Mathematical Sciences, 110, Springer-Verlag, New York, 1995.

[6] E. M. Wright, *A nonlinear difference-differential equation*, J. Reine Angew. Math., 194:66–87, 1955.

[7] G. S. Jones, *On the nonlinear differential-difference equation* $f'(x) = -\alpha f(x-1)1 + f(x)$, J. Math. Anal. Appl., 4:440–469, 1962.

Rigorous Computation of (Un)Stable Manifolds and Connecting Orbits

J. D. Mireles James, Florida Atlantic University

J. D. Mireles James

The study of dynamical systems begins with consideration of basic invariant sets such as equilibria and periodic solutions. After local stability, the next important question is how these basic invariant sets fit together dynamically. Connecting orbits play an important role as they are low dimensional objects which carry global information about the dynamics. This principle is seen at work in the homoclinic tangle theorem of Smale, in traveling wave analysis for reaction diffusion equations, and in Morse homology.

This lecture builds on the validated numerical methods for periodic orbits presented in the lecture of J. B. van den Berg. We will discuss the functional analytic perspective on validated stability analysis for equilibria and periodic orbits as well as validated computation of their local stable/unstable manifolds. With this data in hand we study heteroclinic and homoclinic connecting orbits as solutions of certain projected boundary value problems, and see that these boundary value problems are amenable to an a posteriori analysis very similar to that already discussed for periodic orbits. The discussion will be driven by several application problems including connecting orbits in the Lorenz system and existence of standing and traveling waves.

References

[1] J. B. van den Berg, J. D. Mireles James, J.-P. Lessard, and Konstantin Mischaikow, Rigorous numerics for symmetric connecting orbits: even homoclinics of the Gray-Scott equation. SIAM J. Math. Anal., vol. 43, No. 4, 1557–1594, 2011.

[2] J.-P. Lessard, C. Reinhardt and J. D. Mireles James, Computer assisted proof of transverse saddle-to-saddle connecting orbits for first order vector fields, *Journal of Dynamics and Differential Equations*, vol. 26, Issue 2, 267–313, 2014.

[3] J. B. van den Berg, Andréa Deschênes, J.-P. Lessard and

J. D. Mireles James, Stationary coexistence of hexagons and rolls via rigorous computations, *SIAM J. Applied Dynamical Systems*, vol. 14, No. 2, 942–997, 2015.

[4] R. Castelli, J.-P. Lessard and J. D Mireles James, Parameterization of invariant manifolds for periodic orbits I: efficient numerics via the Floquet normal form, *SIAM J. Applied Dynamical Systems*, vol. 14, no. 1, 132–167, 2015.

[5] J. D. Mireles James, Polynomial approximation of one parameter families of (un)stable manifolds with rigorous computer assisted error bounds, *Indagationes Mathematicae*, vol. 26, Issue 1, 225–265, 2015.

Introduction: General Setup And An Example That Forces Chaos

Jan Bouwe van den Berg, VU University Amsterdam

Studio VU/Riechelle van der Valk.

Jan Bouwe van den Berg

In this lecture the basic concepts of rigorous computing in a dynamical systems context will be outlined. We often simulate dynamics on a computer, or calculate a numerical solution to a partial differential equation. This gives very detailed, stimulating information. However, mathematical insight and impact would be much improved if we can be sure that what we see on the screen genuinely represents a solution of the problem. In particular, rigorous validation of the computations allows such objects to be used as ingredients of theorems.

The past few decades have seen enormous advances in the development of computer assisted proofs in dynamics. In this introductory talk we discuss the basic functional analytic setup underlying the rigorous computational method that is the central topic of this AMS short course. As the central example we will use the problem of finding a particular periodic orbit in a nonlinear ordinary differential equation that describes pattern formation in fluid dynamics. This simple setting keeps technicalities to a minimum. Nevertheless, the rigorous computation of this single periodic orbit implies chaotic behavior via topological arguments (in a sense very similar to "period 3 implies chaos" for interval maps).

References

[1] J. B. van den Berg and J.-P. Lessard, Chaotic braided solutions via rigorous numerics: chaos in the Swift–Hohenberg equation, *SIAM Journal on Applied Dynamical Systems*, 7(3): 988–1031, 2008.

[2] A. Hungria, J.-P. Lessard and J. D. Mireles James, Rigorous numerics for analytic solutions of differential equations: the radii polynomial approach, to appear in *Mathematics of Computation*, 2015.

Bifurcations and an Application in Materials Science

Thomas Wanner, George Mason University

Thomas Wanner

The diblock copolymer model is a fourth-order parabolic partial differential equation which models phase separation with fine structure. The equation is a gradient flow with respect to an extension of the standard van der Waals free energy functional which involves nonlocal interactions, and the long-term dynamical behavior of the diblock copolymer model is described by its finite-dimensional attractor. However, even on one-dimensional domains, the dynamics on the attractor is not fully understood, and rigorous mathematical results on the long-term dynamics of solutions created via phase separation seem to be out of the reach of classical mathematical methods.

In the recent paper [2], it was shown that the location of certain numerically computed bifurcation points in the equilibrium bifurcation diagram can shed light onto this problem. In this lecture we therefore describe how rigorous computational techniques can be used to obtain computer-assisted existence proofs for these bifurcation points. While our presentation is focusing on the diblock copolymer case, the method applies more generally to bifurcation points in infinite-dimensional problems. Particular emphasis is put on fold points and pitchfork bifurcations which are forced through symmetry breaking, as well as the continuation of such bifurcation points in two-parameter problems. The lecture will contain the necessary background material from bifurcation theory, and the approach will be demonstrated using the one-dimensional diblock copolymer equation. Time permitting, we will briefly discuss possible applications in the context of nucleation in a different parabolic partial differential equation known as the Cahn–Morral system.

References

[1] T. Wanner, *Topological analysis of the diblock copolymer equation*, submitted for publication, 2015.

[2] I. Johnson, E. Sander, T. Wanner, *Branch interactions and long-term dynamics for the diblock copolymer model in one dimension*, Discrete and Continuous Dynamical Systems, Series A 33(8), pp. 3671–3705, 2013.

[3] J. P. Desi, H. Edrees, J. J. Price, E. Sander, T. Wanner, *The dynamics of nucleation in stochastic Cahn–Morral systems*, SIAM Journal on Applied Dynamical Systems 10(2), pp. 707–743, 2011.

[4] A. Spence, B. Werner, *Nonsimple turning points and cusps*, IMA Journal of Numerical Analysis 2(4), pp. 413–427, 1982.

[5] B. Werner, A. Spence, *The computation of symmetry-breaking bifurcation points*, SIAM Journal on Numerical Analysis 21(2), pp. 388–399, 1984.

[6] J.-P. Lessard, E. Sander, T. Wanner, *Rigorous continuation of bifurcation points in the diblock copolymer equation*, in preparation, 2015.

Every Calculation an Existence Proof: Towards Automated Rigorous Computing

J. F. Williams, Simon Fraser University

J. F. Williams

For an abstract problem posed as $F(x) = 0$ rigorous computing is, at its core, a strategy to use a computer to evaluate functional analytic bounds numerically. When these bounds are satisfied we prove existence of a true solution in a neighborhood of a numerical candidate. Typically, there is much pencil and paper work to be done to find these bounds required to set up the computation.

In this lecture we will show how to combine algorithms from automatic differentiation with interval arithmetic and the radii polynomial approach to automate both the verification AND construction of the bounds. We will present algorithms to rigorously compute solutions to

a) $f(x) = 0$ for $f: R^n \mapsto R^n$

b) $-\partial u = f(u)$

with the user required to provide little more than an initial guess and the specified function f. The algorithm will then determine the necessary bounds and attempt to verify the solution. Time permitting we will also discuss how to perform continuation on these same problems.

This lecture will assume that participants are familiar (possibly from earlier lectures) with the basics of radii polynomials. We will explain the basics of automatic differentiation, interval arithmetic and explain the framework in which we are using the radii polynomials.

References

1) N. Yamamoto, A numerical verification method for solutions of boundary value problems with local uniqueness by Banach fixed-point theorem, *SIAM J. Numer. Anal.*, **35** (1998), pp. 2004–2013 (electronic).

2) L. B. Rall and G. F. Corliss, An Introduction to Automatic Differentiation in Computational Differentiation: Techniques, Applications, and Tools, *SIAM* (1996).

3) R. E. Moore, R. Baker Kearfott and M. J. Cloud, Introduction to Interval Analysis, *SIAM* (2009).

4) S. M. Rump, INTLAB - INTerval LABoratory, in *Developments in Reliable Computing*, Tibor Csendes, ed., Kluwer Academic Publishers, Dordrecht, 1999.

5) J. B. van den Berg, J.-P. Lessard and K. Mischaikow, Global smooth solution curves using rigorous branch following, *Mathematics of Computation*, **79** (271), 1565–1584, 2010.

6) J. B. van den Berg, C. M. Groothedde, and J. F. Williams, Rigorous Computation of a Radially Symmetric Localized Solution in a Ginzburg–Landau Problem, *SIAM Journal on Applied Dynamical Systems* 14:1, 2015, 423–447.

Registration

There are separate fees to register for this Short Course. Advanced registration fees for members are US$110; nonmembers are US$165, and students/unemployed or emeritus members are US$58. These fees are in effect until December 22, 2015. If you choose to register on-site, the fees for members are US$144; nonmembers are US$195, and students/unemployed or emeritus members are US$79. Advanced registration starts on September 2, 2015. On-site registration will take place on Tuesday, January 5, 2016, at a location to be announced.